Performance Analysis of Protograph LDPC Codes for Nakagami-m Fading Relay Channels

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Abstract

In this paper, we investigate the error performance of the protograph (LDPC) codes over Nakagami-m fading relay channels. We first calculate the decoding thresholds of the protograph codes over such channels with different fading depths (i.e., different values of m) by exploiting the modified protograph extrinsic information transfer (PEXIT) algorithm. Furthermore, based on the PEXIT analysis and using Gaussian approximation, we derive the bit-error-rate (BER) expressions for the error-free (EF) relaying protocol and decode-and-forward (DF) relaying protocol. We finally compare the threshold with the theoretical BER and the simulated BER results of the protograph codes. It reveals that the performance of DF protocol is approximately the same as that of EF protocol. Moreover, the theoretical BER expressions, which are shown to be reasonably consistent with the decoding thresholds and the simulated BERs, are able to evaluate the system performance and predict the decoding threshold with lower complexity as compared to the modified PEXIT algorithm. As a result, this work can facilitate the design of the protograph codes for the wireless communication systems.

Index Terms

Extrinsic information transfer (EXIT) algorithm, Gaussian approximation, Nakagami-m fading, Protograph low-density parity-check (LDPC) code, Relay channels.

I. INTRODUCTION

Spatial diversity is an effective technique to enhance the quality and reliability of wireless communications and can be typically achieved using multiple antennas at a transmitter and/or a receiver. Besides, spatial diversity can also be obtained by the use of relaying. The relay channel, which consists of a source, a relay, and a destination, was probably first proposed in the 70s [1]. Later, Cover and Gamal have further developed the theory of such a channel [2]. Recently, half-duplex relaying [3] has been verified to be relatively more practical in comparison with the full-duplex one due to simpler implementation.

To improve the performance, forward error correction (FEC) codes have been applied to relay channels. As a capacity-approaching code, Low-density parity-check (LDPC) code has been developed to significantly enhance the performance of relay channels for different channel conditions, such as non-fading [3–5] and fading scenarios [6, 7]. Moreover, [8] provided the corresponding analysis of the LDPC-coded relay systems. At the same time, there has been a growing interest in protograph LDPC codes. It was shown in [9–11] that protograph (LDPC) codes not only achieve superior performance but also possess simple structures to realize linear encoding and decoding. For this reason, protograph codes have been used in additive white Gaussian noise (AWGN) relay channels [12], partial response channels [13], and Rayleigh fading channels [14, 15]. However, thus far, its analytical performance for general fading relay channels is not very well understood, which has motivated the results of this paper.

In this paper, we study the error performance of a half-duplex protograph LDPC-coded relay system over Nakagami-m fading channels. The Nakagami-m fading channel is a general type of fading channel that encompasses Rayleigh fading ($m = 1$) and AWGN ($m \to \infty$) channels as special cases. We first use the modified protograph extrinsic information transfer (PEXIT) algorithm [15] to analyze the decoding threshold of the protograph codes with different fading depths (i.e., different values of m). Afterwards, we derive the bit-error-rates (BERs) for the error-free (EF) and the decode-and-forward (DF) relaying protocols exploiting the modified PEXIT algorithm and using Gaussian approximation [16], and find that the BER analytical method has lower computational complexity than that of the modified PEXIT one. Simulated results show that the performance of the DF protocol approaches very close to that of EF, which align well with the analytical results. Furthermore, the results also indicate that the performance of the codes is improved as the fading depth decreases (higher m), but the rate of improvement is reduced simultaneously.

The remainder of this paper is organized as follows. In Section II, the system model over the Nakagami-m fading channel is described. In Section III, we analyze the protograph LDPC coded relay system

using the modified PEXIT algorithm. Moreover, we derive the BER expressions of the protograph codes. Numerical results are carried out in Section IV, and conclusions are given in Section V.

II. SYSTEM MODEL

We consider a two-hop half-duplex relay system model with one source, S, one relay, R, and one destination, D, as shown in Fig. 1. Each transmission period is divided into two time slots, with the first slot being the *broadcast* time slot and the second slot being the *cooperative* time slot. During the first time slot, S broadcasts the codeword to other terminals (including R and D). Then, in the second time slot, R cooperates with S to forward the re-encoded message of S to D while S remains idle.[1] During each transmission period, D stores the received codeword for decoding at the end of the second time slot. Mathematically, the received signals can be written as

$$r_{\mathrm{R1},j} = h_{\mathrm{SR},j} x_j + n_{\mathrm{SR},j}, \tag{1}$$

$$r_{\mathrm{D1},j} = h_{\mathrm{SD},j} x_j + n_{\mathrm{SD},j}, \tag{2}$$

$$r_{\mathrm{D2},j} = h_{\mathrm{RD},j} \hat{x}_j + n_{\mathrm{RD},j}, \tag{3}$$

where, x_j and $\hat{x}_j$ denote the binary-phase-shift-keying (BPSK) modulated signals corresponding to the jth coded bit v_j and re-encoded bit $\hat{v}_j$, respectively; $r_{\mathrm{R1},j}$, $r_{\mathrm{D1},j}$, $r_{\mathrm{D2},j}$ represent the received signals of the jth coded bit at the relay in the 1st time slot, at the destination in the 1st time slot, and at the destination in the 2nd time slot, respectively; $h_{\mathrm{SR},j}$, $h_{\mathrm{SD},j}$, and $h_{\mathrm{RD},j}$ are the mutually independent Nakagami-m fading channel coefficients of the S-R link, the S-D link, and the R-D link, respectively; and $n_{\mathrm{SR},j}$, $n_{\mathrm{SD},j}$, and $n_{\mathrm{RD},j}$ are the AWGN with zero mean and variance of σ_n^2 ($\sigma_n^2 = \frac{N_0}{2}$).

We define the channel gain of the S-R link as $\gamma_{\mathrm{SR},j} = |h_{\mathrm{SR},j}|^2$ (similar expressions will be assumed for other links). Moreover, we assume that the receiver (R or D) knows perfectly the channel state information (CSI) for decoding. At the destination, the signals from the source and relay are combined by a maximum ratio combiner (MRC). To simplify the analysis, we also assume that the distance from S to D is normalized to unity, while the distances from S to R and from R to D are d and $1-d$, respectively.[2] Based on this assumption, we have $h_{\mathrm{SD},j} = \alpha_{\mathrm{SD},j}$, $h_{\mathrm{SR},j} = \frac{\alpha_{\mathrm{SR},j}}{d}$, and $h_{\mathrm{RD},j} = \frac{\alpha_{\mathrm{RD},j}}{1-d}$, where α is the Nakagami-m fading parameter with the probability density function (PDF) expressed as

$$f(x) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m x^{2m-1} \exp\left(-\frac{m}{\Omega}x^2\right), \quad \text{for } m \geq 0.5, \tag{4}$$

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[1]In our model, we assume that the relay and the source adopt the same coding scheme.

[2]Note that the assumption of collinearity of S, R and D will not affect any of the derivation of our results.

where m is the fading depth, $\Omega = \mathbb{E}[\alpha^2] = 1$, $\mathbb{E}[\cdot]$ is the expectation operator, and $\Gamma(\cdot)$ is the Gamma function. In this paper, we focus on ergodic channels, in which the channel fades significantly rapidly such that it varies bit by bit.[3]

III. PERFORMANCE ANALYSIS

In this section, we derive the BER expression of the protograph code in our system with the EF and DF protocols based on the modified PEXIT algorithm and using Gaussian approximation [16]. In the analysis, it is assumed that the all-zero codeword is transmitted and the block length of the code is infinite [17]. The output (extrinsic and *a-posteriori*) log-likelihood-ratio (LLR) messages of Nakagami-m fading channels are approximately following the symmetric Gaussian distribution [18].

Firstly, we review the protograph code [9]. A protograph is a Tanner graph with a relatively small number of nodes. A protograph code is a large protograph (called derived graph) obtained by the "copy-and-permute" operation on a protograph. Accordingly, a code with different block lengths can be produced by different number of times of the "copy-and-permute" operation. A protograph with N variable nodes and M check nodes can be described by a base matrix $\mathbf{B} = (b_{i,j})$ with dimensions $M \times N$, where $b_{i,j}$ denotes the number of edges connecting the variable node v_j to the check node c_i. Since the protograph code always has some punctured variable nodes, we define P_j as the punctured label of v_j ($P_j = 0$ if v_j is punctured; otherwise $P_j = 1$). We also define the following LLRs and mutual information (MI).

LLR value:

- $L_{\text{Av}}(i,j)$ denotes the input (*a-priori*) LLR value of v_j corresponding to c_i on each of the $b_{i,j}$ edges.
- $L_{\text{Ac}}(i,j)$ denotes the input (*a-priori*) LLR value of c_i corresponding to v_j on each of the $b_{i,j}$ edges.
- $L_{\text{Ev}}(i,j)$ denotes the output (extrinsic) LLR value passing from v_j to the c_i.
- $L_{\text{Ec}}(i,j)$ denotes the output (extrinsic) LLR value passing from c_i to the v_j.
- $L_{\text{app}}(j)$ denotes the *a-posteriori* LLR value of v_j.

MI:

- $I_{\text{Av}}(i,j)$ denotes the *a-priori* MI between $L_{\text{Av}}(i,j)$ and the corresponding coded bit v_j.
- $I_{\text{Ac}}(i,j)$ denotes the *a-priori* MI between $L_{\text{Ac}}(i,j)$ and the corresponding coded bit v_j.
- $I_{\text{Ev}}(i,j)$ denotes the extrinsic MI between $L_{\text{Ev}}(i,j)$ and the corresponding coded bit v_j.
- $I_{\text{Ec}}(i,j)$ denotes the extrinsic MI between $L_{\text{Ec}}(i,j)$ and the corresponding coded bit v_j.

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[3]Although we consider the ergodic (i.e., fast fading) scenario here, the results in this paper are also applicable for the quasi-static case (where the fading parameter of each link is kept to be constant for a code block).

- $I_{\mathrm{app}}(j)$ denotes the *a-posteriori* MI between $L_{\mathrm{app}}(j)$ and the corresponding coded bit v_j.

Note that the subscripts "SR", "SD", "RD", and "D" are used to denote the S-R link, the S-D link, the R-D link, and the destination, respectively. For example, $L_{\mathrm{SR-Av}}(i,j)$ represents the a-priori LLR value of v_j corresponding to c_i on each of the $b_{i,j}$ edges of the S-R link.

A. BER of EF Relaying Protocol

At the destination receiver, the output of MRC corresponding to the jth coded bit is given by

$$y_{\mathrm{D},j} = h_{\mathrm{SD},j}r_{\mathrm{D1},j} + h_{\mathrm{RD},j}r_{\mathrm{D2},j}. \tag{5}$$

Afterwards, the initial LLR value $L_{\mathrm{D-ch},j}$ of the jth coded bit is calculated as

$$
\begin{aligned}
L_{\mathrm{D-ch},j} &= \ln\frac{\mathrm{Pr}(v_j = 0|y_{\mathrm{D},j},\mathbf{h}_j)}{\mathrm{Pr}(v_j = 1|y_{\mathrm{D},j},\mathbf{h}_j)} = \ln\frac{\mathrm{Pr}(x_j = +1|y_{\mathrm{D},j},\mathbf{h}_j)}{\mathrm{Pr}(x_j = -1|y_{\mathrm{D},j},\mathbf{h}_j)} \\
&= \frac{2y_{\mathrm{D},j}}{\sigma_n^2} = \frac{2}{\sigma_n^2}(h_{\mathrm{SD},j}r_{\mathrm{D1},j} + h_{\mathrm{RD},j}r_{\mathrm{D2},j}) \\
&= \frac{2}{\sigma_n^2}\left[h_{\mathrm{SD},j}(h_{\mathrm{SD},j}x_j + n_{\mathrm{SD},j}) + h_{\mathrm{RD},j}(h_{\mathrm{RD},j}\hat{x}_j + n_{\mathrm{RD},j})\right] \\
&= \frac{2}{\sigma_n^2}\left[|h_{\mathrm{SD},j}|^2 x_j + |h_{\mathrm{RD},j}|^2\hat{x}_j + h_{\mathrm{SD},j}n_{\mathrm{SD},j} + h_{\mathrm{RD},j}n_{\mathrm{RD},j}\right] \\
&= \frac{2}{\sigma_n^2}\left[(\gamma_{\mathrm{SD},j} + \gamma_{\mathrm{RD},j})x_j + h_{\mathrm{SD},j}n_{\mathrm{SD},j} + h_{\mathrm{RD},j}n_{\mathrm{RD},j}\right],
\end{aligned} \tag{6}
$$

where $\mathrm{Pr}(\cdot)$ denotes the probability function, $\mathbf{h}_j = \{h_{\mathrm{SR},j}, h_{\mathrm{SD},j}, h_{\mathrm{RD},j}\}$ is the channel vector, $\gamma_{\mathrm{SD},j} = |h_{\mathrm{SD},j}|^2 = \alpha_{\mathrm{SD},j}^2$, $\gamma_{\mathrm{RD},j} = |h_{\mathrm{RD},j}|^2 = \frac{\alpha_{\mathrm{RD},j}^2}{(1-d)^2}$, and $x_j = \hat{x}_j$ (error-free decoding at the relay).

Based on the all-zero codeword assumption ($x_j = +1$), given a fixed channel realization (a fixed fading vector $\mathbf{h}_j$), we can easily obtain the expectation and the variance of $L_{\mathrm{D-ch},j}$, resulting in

$$\mathbb{E}[L_{\mathrm{D-ch},j}] = \frac{2}{\sigma_n^2}(\gamma_{\mathrm{SD},j} + \gamma_{\mathrm{RD},j}) = \frac{2}{\sigma_n^2}\lambda_{\mathrm{D},j}, \tag{7}$$

$$\mathrm{var}[L_{\mathrm{D-ch},j}] = \frac{4}{\sigma_n^4}(\gamma_{\mathrm{SD},j}\sigma_n^2 + \gamma_{\mathrm{RD},j}\sigma_n^2) = \frac{4}{\sigma_n^2}\lambda_{\mathrm{D},j}, \tag{8}$$

where $\lambda_{\mathrm{D},j}$ is the short-hand notation of $\gamma_{\mathrm{SD},j} + \gamma_{\mathrm{RD},j}$. As seen from (7) and (8), $L_{\mathrm{D-ch},j}$ follows the symmetric Gaussian distribution, i.e., $L_{\mathrm{D-ch},j} \sim \mathcal{N}(\frac{2}{\sigma_n^2}\lambda_{\mathrm{D},j}, \frac{4}{\sigma_n^2}\lambda_{\mathrm{D},j})$. Subsequently, substituting $\sigma_n^2 = \frac{1}{2R(E_b'/N_0)}$ into (8) ($E_b' = \frac{E_b}{2}$, the normalized factor $1/2$ is used to keep the total energy per transmitted symbol to be E_s) and considering the punctured label P_j, we have

$$\mathrm{var}[L_{\mathrm{D-ch},j}] = 8RP_j\lambda_{\mathrm{D},j}(E_b'/N_0) = 4RP_j\lambda_{\mathrm{D},j}(E_b/N_0). \tag{9}$$

Thus, the modified PEXIT algorithm [15] can be applied to our system by using the expression (9).[4]

For the qth ($q = 1, 2 \ldots, Q$, where Q is the total number of channel realizations[5]) channel realization $h_{q,j}$, the a-posteriori LLR of v_j during the $(t+1)$th iteration is written as

$$L_{\text{D−app},q}^{t+1}(j) = \sum_{i=1}^{M} b_{i,j} L_{\text{D−Av},q}^{t+1}(i,j) + L_{\text{D−ch},q,j}. \tag{10}$$

The expected value of $L_{\text{D−Av},q}^{t+1}(i,j)$ is then expressed by

$$\bar{L}_{\text{D−app}}^{t+1}(j) = \mathbb{E}[L_{\text{D−app},q}^{t+1}(j)] = \frac{1}{Q} \sum_{q=1}^{Q} L_{\text{D−app},q}^{t+1}(j). \tag{11}$$

As the output LLR values, i.e., the extrinsic LLR and a-posteriori LLR, are approximated to follow a symmetric Gaussian distribution, we can evaluate the variance of $\bar{L}_{\text{D−app}}^{t+1}(j)$ as

$$\text{var}[\bar{L}_{\text{D−app}}^{t+1}(j)] = \left\{ J^{-1}(\bar{I}_{\text{D−app}}^{t+1}(j)) \right\}^2 = \left\{ J^{-1}(\mathbb{E}[I_{\text{D−app},q}^{t+1}(j)]) \right\}^2, \tag{12}$$

where $\bar{I}_{\text{D−app}}^{t+1}(j) = \mathbb{E}[I_{\text{D−app},q}^{t+1}(j)] = \frac{1}{Q} \sum_{q=1}^{Q} I_{\text{D−app},q}^{t+1}(j)$ is the expected value of $I_{\text{D−app},q}^{t+1}(j)$ over all channel realizations, the expression of the a-posteriori MI $I_{\text{D−app},q}^{t+1}(j)$ is obtained as [15, (21)], denoted by $I_{\text{D−app},q}^{t+1}(j) = F(I_{\text{D−Av},q}^{t+1}(1,j), \ldots, I_{\text{D−Av},q}^{t+1}(M,j); \text{var}[L_{\text{D−ch},q,j}])$, and $J(\sigma_{\text{ch}})$ is the MI between a BPSK modulated bit and its LLR value $L_{\text{ch}} \sim \mathcal{N}(\frac{\sigma_{\text{ch}}^2}{2}, \sigma_{\text{ch}}^2)$ over an AWGN channel, represented as [17]

$$J(\sigma_{\text{ch}}) = 1 - \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{(\xi - \sigma_{\text{ch}}^2/2)^2}{2\sigma_{\text{ch}}^2}\right)}{\sqrt{2\pi\sigma_{\text{ch}}^2}} \log_2\left[1 + \exp(-\xi)\right] d\xi. \tag{13}$$

The corresponding inverse function is given by [17]

$$J^{-1}(x) = \begin{cases} \eta_1 x^2 + \eta_2 x + \eta_3 \sqrt{x} & \text{if } 0 \leq x \leq 0.3646, \\ \eta_4 \ln[\eta_5(1-x)] + \eta_6 x & \text{otherwise,} \end{cases} \tag{14}$$

where $\eta_1 = 1.09542$, $\eta_2 = 0.214217$, $\eta_3 = 2.33737$, $\eta_4 = -0.706692$, $\eta_5 = 0.386013$ and $\eta_6 = 1.75017$. With the help of (12)–(14), the BER of the jth variable node after t iterations is evaluated by

$$\begin{aligned} P_{\text{D−}b}^{t+1}(j) &= \frac{1}{2} \Pr(\bar{L}_{\text{D−app}}^{t+1}(j) < 0 | x_j = +1) + \frac{1}{2} \Pr(\bar{L}_{\text{D−app}}^{t+1}(j) \geq 0 | x_j = -1) \\ &= \frac{1}{2}\text{erfc}\left(\frac{\mathbb{E}[\bar{L}_{\text{D−app}}^{t+1}(j) | x_j = +1]}{\sqrt{2\text{var}[\bar{L}_{\text{D−app}}^{t+1}(j) | x_j = +1]}}\right) = \frac{1}{2}\text{erfc}\left(\frac{\text{var}[\bar{L}_{\text{D−app}}^{t+1}(j) | x_j = +1]/2}{\sqrt{2\text{var}[\bar{L}_{\text{D−app}}^{t+1}(j) | x_j = +1]}}\right) \\ &= \frac{1}{2}\text{erfc}\left(\frac{\sqrt{\text{var}[\bar{L}_{\text{D−app}}^{t+1}(j) | x_j = +1]}}{2\sqrt{2}}\right) = \frac{1}{2}\text{erfc}\left(\frac{J^{-1}(\bar{I}_{\text{D−app}}^{t+1}(j))}{2\sqrt{2}}\right), \end{aligned} \tag{15}$$

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[4]According to [15], the maximum iteration number of the modified PEXIT algorithm $T_{\max}^{\text{P}}$ should be large enough in order to ensure the complete convergence of the decoder, i.e., $T_{\max}^{\text{P}} \geq 500$.

[5]To ensure the accuracy of our derived BER expressions, Q should be set to a sufficiently large integer, i.e., $Q \geq 10^5$.

where $\mathrm{erfc}(\cdot)$ is the complementary error function, defined as $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\tau^2} \mathrm{d}\tau$.

Finally, the averaged BER of a protograph code after t iterations is written as

$$P_{\mathrm{EF}-b}^{t+1} = P_{\mathrm{D}-b}^{t+1} = \frac{1}{N} \sum_{j=1}^{N} P_{\mathrm{D}-b}^{t+1}(j) = \frac{1}{2N} \sum_{j=1}^{N} \mathrm{erfc}\left(\frac{J^{-1}(\bar{I}_{\mathrm{D-app}}^{t+1}(j))}{2\sqrt{2}} \right). \tag{16}$$

B. BER of DF Relaying Protocol

In DF, if the relay can decode received signals correctly, it re-encodes the decoded information and then forwards them to the destination; otherwise, it does not send message or remains idle. Consequently, the BER of the DF protocol of the jth variable node after t iterations can be found as

$$P_{\mathrm{DF}-b}^{t+1}(j) = P_{\mathrm{SR}-b}^{t+1}(j) P_{\mathrm{SD}-b}^{t+1}(j) + \left[1 - P_{\mathrm{SR}-b}^{t+1}(j)\right] P_{\mathrm{D}-b}^{t+1}(j), \tag{17}$$

where $P_{\mathrm{SR}-b}^{t+1}(j)$, $P_{\mathrm{SD}-b}^{t+1}(j)$, and $P_{\mathrm{D}-b}^{t+1}(j)$ are the corresponding BERs of the jth variable node after t iterations at the relay receiver with the signal from source, at the destination receiver with one signal from the source, and at the destination receiver with two signals both from the source and relay (i.e., BER of the EF protocol), respectively.

The initial LLR of the jth variable node of the S-R link is expressed by [18]

$$L_{\mathrm{SR-ch},j} = \frac{2h_{\mathrm{SR},j} r_{\mathrm{R1},j}}{\sigma_n^2} = \frac{2h_{\mathrm{SR},j}}{\sigma_n^2}(h_{\mathrm{SR},j} x_j + n_{\mathrm{SR},j}) = \frac{2}{\sigma_n^2}(\gamma_{\mathrm{SR},j} x_j + h_{\mathrm{SR},j} n_{\mathrm{SR},j}). \tag{18}$$

In (18), we have $\mathbb{E}[L_{\mathrm{SR-ch},j}] = \frac{2\gamma_{\mathrm{SR},j}}{\sigma_n^2}$ and $\mathrm{var}[L_{\mathrm{SR-ch},j}] = \frac{4\gamma_{\mathrm{SR},j}}{\sigma_n^2}$. Hence, during the $(t+1)$th iteration, the a-posterior MI of the jth variable for the qth ($q = 1, 2, \ldots, Q$) channel realization and the corresponding expected value are respectively given by

$$I_{\mathrm{SR-app},q}^{t+1}(j) = F(I_{\mathrm{SR-Av},q}^{t+1}(1,j), \ldots, I_{\mathrm{SR-Av},q}^{t+1}(M,j); \mathrm{var}[L_{\mathrm{SR-ch},q,j}]), \tag{19}$$

$$\bar{I}_{\mathrm{SR-app}}^{t+1}(j) = \mathbb{E}[I_{\mathrm{SR-app},q}^{t+1}(j)] = \frac{1}{Q} \sum_{q=1}^{Q} I_{\mathrm{SR-app},q}^{t+1}(j). \tag{20}$$

Further, the variance of the expected a-posterior LLR associated with (20) is yielded in terms of the Gaussian assumption

$$\mathrm{var}[\bar{L}_{\mathrm{SR-app}}^{t+1}(j)] = \left\{ J^{-1}(\bar{I}_{\mathrm{SR-app}}^{t+1}(j)) \right\}^2. \tag{21}$$

Note that $\mathbb{E}[\bar{L}_{\mathrm{SR-app}}^{t+1}(j)] = \mathrm{var}[\bar{L}_{\mathrm{SR-app}}^{t+1}(j)]/2$.

Using (21), we can therefore obtain the BER of the jth variable after t iterations as

$$P_{\mathrm{SR}-b}^{t+1}(j) = \frac{1}{2}\mathrm{erfc}\left(\frac{\mathbb{E}[\bar{L}_{\mathrm{SR-app}}^{t+1}(j)|x_j = +1]}{\sqrt{2\mathrm{var}[\bar{L}_{\mathrm{SR-app}}^{t+1}(j)|x_j = +1]}} \right) = \frac{1}{2}\mathrm{erfc}\left(\frac{J^{-1}(\bar{I}_{\mathrm{SR-app}}^{t+1}(j))}{2\sqrt{2}} \right). \tag{22}$$

Likewise, the BER of the jth variable after t iterations of the S-D link can be written as

$$P_{\mathrm{SD}-b}^{t+1}(j) = \frac{1}{2}\mathrm{erfc}\left(\frac{J^{-1}(\bar{I}_{\mathrm{SD-app}}^{t+1}(j))}{2\sqrt{2}}\right) \tag{23}$$

with the parameters subjected to

$$\begin{cases} \bar{I}_{\mathrm{SD-app}}^{t+1}(j) = \mathbb{E}[I_{\mathrm{SD-app},q}^{t+1}(j)] = \dfrac{1}{Q}\sum_{q=1}^{Q} I_{\mathrm{SD-app},q}^{t+1}(j) \\[2mm] I_{\mathrm{SD-app},q}^{t+1}(j) = F(I_{\mathrm{SD-Av},q}^{t+1}(1,j),\ldots,I_{\mathrm{SD-Av},q}^{t+1}(M,j);\mathrm{var}[L_{\mathrm{SD-ch},q,j}]) \\[2mm] \mathrm{var}[L_{\mathrm{SD-ch},q,j}] = \dfrac{4\gamma_{\mathrm{SD},q,j}}{\sigma_n^2}, \end{cases} \tag{24}$$

where $L_{\mathrm{SD-ch},q,j}$ is the initial LLR of v_j for the qth S-D link realization and $\gamma_{\mathrm{SD},q,j} = |h_{\mathrm{SD},q,j}|^2$.

Substituting (15), (22) and (23) into (17), the BER of the jth variable node with DF after t iterations can be formulated. Finally, we get the averaged BER with DF protocol after t iterations as

$$P_{\mathrm{DF}-b}^{t+1} = \frac{1}{N}\sum_{j=1}^{N} P_{\mathrm{DF}-b}^{t+1}(j). \tag{25}$$

Note also that

- The maximum number of iterations of the theoretical BER analysis $T_{\max}$ ($t = T_{\max}$) equals to that of the simulations ($T_{\max}$ is always much smaller than $T_{\max}^{\mathrm{P}}$). In this paper, we set $T_{\max} = 100$ as in [15, 20]. Therefore, its computational complexity can be reduced by $\frac{(T_{\max}^{\mathrm{P}}-T_{\max})}{T_{\max}^{\mathrm{P}}}$ as compared to the modified PEXIT algorithm.

- The BER analysis can be used to evaluate the performance of the protograph codes for any E_b/N_0 and $T_{\max}$, while the modified PEXIT algorithm can only be exploited to derive the E_b/N_0 threshold above which an arbitrarily small BER can be achieved (i.e., all the code blocks can be successfully decoded) for a sufficiently large $T_{\max}^{\mathrm{P}}$.

IV. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we firstly analyze the decoding threshold of two typical protograph codes, i.e., the AR3A and AR4JA codes, over Nakagami-m fading relay channels utilizing the modified PEXIT algorithm [15]. Then, we compare the decoding thresholds, the theoretical BER and simulated BER results of the two protograph codes. For all the following results, the distance of S-R link d is set to 0.4.

The AR3A and AR4JA codes, which have been proposed by Jet Propulsion Laboratory [10, 11], can respectively accomplish excellent performance in the low SNR region and the high SNR region over the

AWGN channel. The corresponding base matrices of these two codes with a code rate of $R = \frac{n+1}{n+2}$, denoted by $\mathbf{B}_{A3}$ and $\mathbf{B}_{A4}$, respectively, are given by

$$\mathbf{B}_{A3} = \begin{pmatrix} 1\,2\,1\,0\,0\,0\,\overbrace{0\,0\,\cdots\,0\,0}^{2n} \\ 0\,2\,1\,1\,1\,2\,1\,\cdots\,2\,1 \\ 0\,1\,2\,1\,1\,1\,2\,\cdots\,1\,2 \end{pmatrix}, \tag{26}$$

$$\mathbf{B}_{A4} = \begin{pmatrix} 1\,2\,0\,0\,0\,0\,\overbrace{0\,0\,\cdots\,0\,0}^{2n} \\ 0\,3\,1\,1\,1\,3\,1\,\cdots\,3\,1 \\ 0\,1\,2\,2\,1\,1\,3\,\cdots\,1\,3 \end{pmatrix}. \tag{27}$$

In (26) and (27), the jth column corresponds to the jth variable node and the ith row refers to the ith check node. The variable nodes corresponding to the second columns of the two matrices are punctured.

A. Decoding Threshold Analysis

We calculate the decoding thresholds of the AR3A and AR4JA codes with different code rates and different fading depths using the modified PEXIT algorithm [15] and show the result in Table I.[6] As seen from this table, the threshold of the AR3A code is lower than that of the AR4JA code for a fixed code rate and fading depth (fixed R and m). For instance, the thresholds of the AR3A code and the AR4JA code are 0.575 dB and 0.722 dB, respectively, with parameters $R = 4/5$ and $m = 2$. Moreover, the threshold is reduced as the fading depth decreases (higher m) for both the two codes and hence the error performance should be improved in the waterfall region. However, the decrease of the threshold is reduced when m becomes larger, indicating that the rate of performance improvement is reduced.

B. BER Performance

In the sequel, we present the simulated results of the AR3A and AR4JA codes with a code rate of $R = 4/5$ and compare their simulated and theoretical BER curves with the decoding thresholds. We denote N_{p}, K_{p}, and L_{p} as the block length, the information length, and the punctured length of the code, respectively. Unless specified otherwise, in the simulations, it has been assumed that the decoder performs a maximum of 100 iterations for each code block with parameters $[N_{\mathrm{p}}, K_{\mathrm{p}}, L_{\mathrm{p}}] = [5632, 4096, 512]$.

In Fig. 2, we show the BER results (vs. E_b/N_0) of the AR4JA and AR3A codes over Nakagami-fading relay channels with DF and EF protocols. The fading depth m is set to 2. As seen, the error performance

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[6]For the threshold analysis, we only consider the perfect S-R channel (i.e., EF), as is typical in many practical cases [19].

of the codes with the DF protocol approaches closely to that with EF, which suggests that the relay can decode most of the received code blocks successfully. Moreover, the AR3A code outperforms the AR4JA codes for the range of E_b/N_0 under study. At a BER of 10^{-5}, the AR3A code achieves a gain approximately 0.2 dB as compared to the AR4JA code both for the DF and EF protocols. In the same figure, consider the AR3A code with DF protocol at a BER of 10^{-5}, the theoretical BER result is in good agreement with the PEXIT threshold within 0.2 dB. Moreover, the corresponding simulated curve has another gap about 0.7 dB to the theoretical one, which is reasonably consistent with the results in [16, 20]. It is because our theoretical BER formulas are derived subjecting to the infinite-block length assumption.

Fig. 3 presents the BER results (vs. E_b/N_0) of the AR3A code in the EF relay systems for different fading depths ($m = 1, 2, 3$, and 4). As E_b/N_0 increases steadily, both the two codes start to perform better for a larger value of m in terms of the thresholds, the theoretical and simulated BER curves. Nevertheless, the improved gain is reduced as m increases. For example, at a BER of 10^{-5}, the improved gains are about 1.2 dB, 0.45 dB, and 0.25 dB as m increases from 1 to 2, 2 to 3, and 3 to 4, respectively. Moreover, it can be observed that gaps between the theoretical BERs and the simulated results are around 0.8 dB, 0.7 dB, 0.55 dB, and 0.5 dB for $m = 1, 2, 3$, and 4, respectively. Simulations have also been performed for the AR4JA code and with the DF protocol, and similar observations are obtained.

V. CONCLUSIONS

The performance of the protograph LDPC codes over Nakagami-m fading relay channels has been studied. The BER expressions for the protograph codes with DF and EF relaying protocols have been derived using the modified PEXIT algorithm and Gaussian approximation. The decoding threshold, the theoretical BER and the simulated BER results have shown that the error performance of the DF protocol is very close to that of EF, which suggests that the relay can decode most received codewords correctly. The differences between the threshold and the theoretical BER and between the theoretical BER and the simulated BER have been found to be around $0.15 - 0.2$ dB and $0.5 - 0.8$ dB, respectively, showing a reasonable consistence. Consequently, our analytical expressions not only can provide a good approximation of the system performance for a large-block length but also predict accurately the decoding threshold more efficiently in comparison with the modified PEXIT algorithm.

VI. ACKNOWLEDGMENTS

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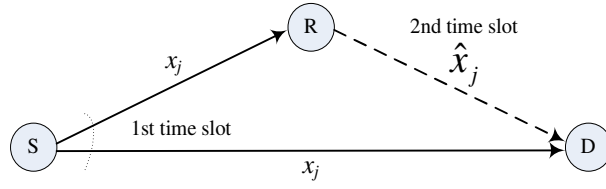LIST OF FIGURES

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Fig. 1. The system model of the protograph LDPC-coded relay system over fading channels.
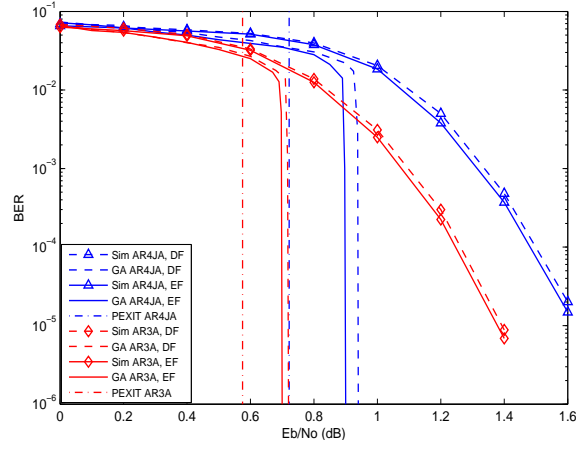

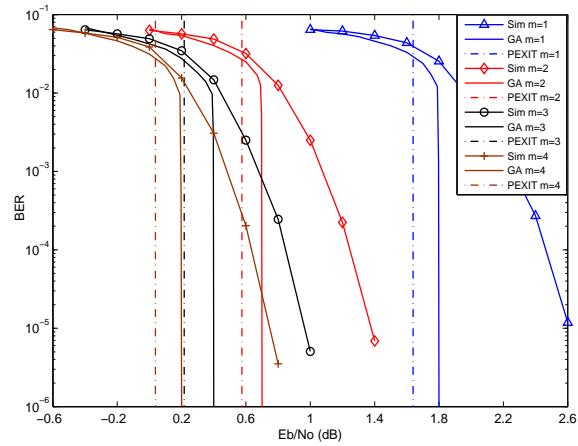Fig. 2. BER results of AR3A and AR4JA codes over Nakagami-fading relay channels with the fading depth $m = 2$.

Fig. 3. BER results of the AR3A code over Nakagami-fading relay channels with different fading depths ($m = 1, 2, 3$, and 4).

TABLE I

DECODING THRESHOLDS $(E_b/N_0)_{\text{th}}$ (dB) OF THE AR3A CODE AND AR4JA CODE WITH DIFFERENT CODE RATES OVER NAKAGAMI-FADING RELAY CHANNELS WITH DIFFERENT FADING DEPTHS $m = 1, 2, 3$, AND 4.

Code Rate	AR3A code				AR4JA code			
	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$1/2\,(n = 0)$	-1.345	-1.825	-1.983	-2.056	-1.187	-1.675	-1.828	-1.895
$2/3\,(n = 1)$	0.004	-0.759	-0.994	-1.098	0.162	-0.566	-0.823	-0.918
$3/4\,(n = 2)$	0.890	-0.016	-0.316	-0.472	1.128	0.185	-0.136	-0.292
$4/5\,(n = 3)$	1.639	0.575	0.216	0.037	1.805	0.722	0.382	0.192
$5/6\,(n = 4)$	2.235	1.042	0.653	0.454	2.392	1.164	0.785	0.576
$6/7\,(n = 5)$	2.732	1.404	0.991	0.778	2.855	1.533	1.092	0.887
$7/8\,(n = 6)$	3.158	1.748	1.278	1.044	3.284	1.852	1.379	1.150